

GROWTH ESTIMATES FOR A CLASS OF SUBHARMONIC FUNCTIONS IN A HALF PLANE *

PAN GUOSHUANG^{1,2} AND DENG GUANTIE^{1,**}

ABSTRACT. A class of subharmonic functions represented by the modified kernels are proved to have the growth estimates $u(z) = o(y^{1-\alpha}|z|^{m+\alpha})$ at infinity in the upper half plane \mathbf{C}_+ , which generalizes the growth properties of analytic functions and harmonic functions.

1. INTRODUCTION AND MAIN THEOREM

Let \mathbf{C} denote the complex plane with points $z = x + iy$, where $x, y \in \mathbf{R}$. The boundary and closure of an open Ω of \mathbf{C} are denoted by $\partial\Omega$ and $\overline{\Omega}$ respectively. The upper half-plane \mathbf{C}_+ is the set $\mathbf{C}_+ = \{z = x + iy \in \mathbf{C} : y > 0\}$, whose boundary is $\partial\mathbf{C}_+$. We write $B(z, \rho)$ and $\partial B(z, \rho)$ for the open ball and the sphere of radius ρ centered at z in \mathbf{C} . We identify $\partial\mathbf{C}_+$ with \mathbf{R} .

For $z \in \mathbf{C} \setminus \{0\}$, let ([3])

$$E(z) = (2\pi)^{-1} \log |z|$$

where $|z|$ is the Euclidean norm. We know that E is locally integrable in \mathbf{C} .

We define the Green function $G(z, \zeta)$ for the upper half plane \mathbf{C}_+ by ([3])

$$G(z, \zeta) = E(z - \zeta) - E(z - \bar{\zeta}) \quad z, \zeta \in \overline{\mathbf{C}_+}, \quad z \neq \zeta. \quad (1.1)$$

We define the Poisson kernel $P(z, \xi)$ when $z \in \mathbf{C}_+$ and $\xi \in \partial\mathbf{C}_+$ by

$$P(z, \xi) = -\frac{\partial G(z, \zeta)}{\partial \eta} \Big|_{\eta=0} = \frac{y}{\pi|z - \xi|^2}.$$

The Dirichlet problem of upper half plane is to find a function u satisfying

$$u \in C^2(\mathbf{C}_+),$$

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**Corresponding author.

$$\Delta u = 0, z \in \mathbf{C}_+,$$

$$\lim_{z \rightarrow x} u(z) = f(x) \text{ nontangentially a.e. } x \in \partial \mathbf{C}_+,$$

where f is a measurable function of \mathbf{R} . The Poisson integral of the upper half plane is defined by

$$v(z) = P[f](z) = \int_{\mathbf{R}} P(z, \xi) f(\xi) d\xi. \quad (1.2)$$

We have know that, the Poisson integral $P[f]$ exists if

$$\int_{\mathbf{R}} \frac{|f(\xi)|}{1 + |\xi|^2} d\xi < \infty. \quad (1.3)$$

(see [4] and [5]) In this paper, we will consider measurable functions f in \mathbf{R} satisfying

$$\int_{\mathbf{R}} \frac{|f(\xi)|}{1 + |\xi|^{2+m}} d\xi < \infty, \quad (1.4)$$

where m is a natural number. To obtain a solution of Dirichlet problem for the boundary date f , we use the following modified functions defined by

$$E_m(z - \zeta) = \begin{cases} E(z - \zeta) & \text{when } |\zeta| \leq 1, \\ E(z - \zeta) - \frac{1}{2\pi} \Re(\log \zeta - \sum_{k=1}^{m-1} \frac{z^k}{k \zeta^k}) & \text{when } |\zeta| > 1. \end{cases}$$

Then we can define modified Green function $G_m(z, \zeta)$ and the modified Poisson kernel $P_m(z, \xi)$ by

$$G_m(z, \zeta) = E_{m+1}(z - \zeta) - E_{m+1}(z - \bar{\zeta}) \quad z, \zeta \in \overline{\mathbf{C}_+}, z \neq \zeta; \quad (1.5)$$

$$P_m(z, \xi) = \begin{cases} P(z, \xi) & \text{when } |\xi| \leq 1, \\ P(z, \xi) - \frac{1}{\pi} \Im \sum_{k=0}^m \frac{z^k}{\xi^{1+k}} & \text{when } |\xi| > 1. \end{cases} \quad (1.6)$$

where $z = x + iy, \zeta = \xi + i\eta$.

Hayman([1]) has proved the following result:

Theorem A Let f be a measurable function in \mathbf{R} satisfying (1.3), let μ be a Borel positive measure satisfying

$$\int_{\mathbf{C}_+} \frac{\eta}{1 + |\zeta|^2} d\mu(\zeta) < \infty.$$

Write the subharmonic function

$$u(z) = v(z) + h(z), \quad z \in \mathbf{C}_+$$

where $v(z)$ be the harmonic function defined by (1.2), $h(z)$ is defined by

$$h(z) = \int_{\mathbf{C}_+} G(z, \zeta) d\mu(\zeta)$$

and $G(z, \zeta)$ is defined by (1.1). Then there exists $z_j \in \mathbf{C}_+$, $\rho_j > 0$, such that

$$\sum_{j=1}^{\infty} \frac{\rho_j}{|z_j|} < \infty$$

holds and

$$u(z) = o(|z|) \quad \text{as } |z| \rightarrow \infty$$

holds in $\mathbf{C}_+ - G$, where $G = \bigcup_{j=1}^{\infty} B(z_j, \rho_j)$.

Our aim in this paper is to establish the following theorems.

Theorem 1 Let f be a measurable function in \mathbf{R} satisfying (1.4), and $0 < \alpha \leq 2$. Let $v(z)$ be the harmonic function defined by

$$v(z) = \int_{\mathbf{R}} P_m(z, \xi) f(\xi) d\xi \quad z \in \mathbf{C}_+ \quad (1.7)$$

where $P_m(z, \xi)$ is defined by (1.6). Then there exists $z_j \in \mathbf{C}_+$, $\rho_j > 0$, such that

$$\sum_{j=1}^{\infty} \frac{\rho_j^{2-\alpha}}{|z_j|^{2-\alpha}} < \infty \quad (1.8)$$

holds and

$$v(z) = o(y^{1-\alpha}|z|^{m+\alpha}) \quad \text{as } |z| \rightarrow \infty \quad (1.9)$$

holds in $\mathbf{C}_+ - G$, where $G = \bigcup_{j=1}^{\infty} B(z_j, \rho_j)$.

Remark 1 If $\alpha = 2$, then (1.8) is a finite sum, the set G is a bounded set, so (1.9) holds in \mathbf{C}_+ .

Next, we will generalize Theorem 1 to subharmonic functions.

Theorem 2 Let f be a measurable function in \mathbf{R} satisfying (1.4), let μ be a Borel positive measure satisfying

$$\int_{\mathbf{C}_+} \frac{\eta}{1 + |\zeta|^{2+m}} d\mu(\zeta) < \infty.$$

Write the subharmonic function

$$u(z) = v(z) + h(z), \quad z \in \mathbf{C}_+$$

where $v(z)$ be the harmonic function defined by (1.7), $h(z)$ is defined by

$$h(z) = \int_{\mathbf{C}_+} G_m(z, \zeta) d\mu(\zeta)$$

and $G_m(z, \zeta)$ is defined by (1.5). Then there exists $z_j \in \mathbf{C}_+$, $\rho_j > 0$, such that (1.8) holds and

$$u(z) = o(y^{1-\alpha}|z|^{m+\alpha}) \quad \text{as } |z| \rightarrow \infty \quad (1.10)$$

holds in $\mathbf{C}_+ - G$, where $G = \bigcup_{j=1}^{\infty} B(z_j, \rho_j)$ and $0 < \alpha < 2$.

Remark 2 If $\alpha = 1, m = 0$, this is just the result of Hamman, so our result (1.10) is the generalization of Theorem A.

2. PROOF OF THEOREM

Let μ be a positive Borel measure in \mathbf{C} , $\beta \geq 0$, the maximal function $M(d\mu)(z)$ of order β is defined by

$$M(d\mu)(z) = \sup_{0 < r < \infty} \frac{\mu(B(z, r))}{r^\beta},$$

then the maximal function $M(d\mu)(z) : \mathbf{C} \rightarrow [0, \infty)$ is semicontinuous, hence measurable. To see this, $\forall \lambda > 0$, let $D(\lambda) = \{z \in \mathbf{C} : M(d\mu)(z) > \lambda\}$. Fix $z \in D(\lambda)$, then $\exists r > 0$ such that $\mu(B(z, r)) > tr^\beta$ for some $t > \lambda$, and $\exists \delta > 0$ satisfying $(r + \delta)^\beta < \frac{tr^\beta}{\lambda}$. If $|\zeta - z| < \delta$, then $B(\zeta, r + \delta) \supset B(z, r)$, therefore $\mu(B(\zeta, r + \delta)) \geq tr^\beta = t(\frac{r}{r + \delta})^\beta (r + \delta)^\beta > \lambda(r + \delta)^\beta$. Thus $B(z, \delta) \subset D(\lambda)$. This proves that $D(\lambda)$ is open for each $\lambda > 0$.

In order to obtain the result, we need these lemmas below:

Lemma 1 Let μ be a positive Borel measure in \mathbf{C} , $\beta \geq 0$, $\mu(\mathbf{C}) < \infty$, $\forall \lambda \geq 5^\beta \mu(\mathbf{C})$, set

$$E(\lambda) = \{z \in \mathbf{C} : |z| \geq 2, M(d\mu)(z) > \frac{\lambda}{|z|^\beta}\}$$

then $\exists z_j \in E(\lambda)$, $\rho_j > 0$, $j = 1, 2, \dots$, such that

$$E(\lambda) \subset \bigcup_{j=1}^{\infty} B(z_j, \rho_j) \quad (2.1)$$

and

$$\sum_{j=1}^{\infty} \frac{\rho_j^\beta}{|z_j|^\beta} \leq \frac{3\mu(\mathbf{C})5^\beta}{\lambda}. \quad (2.2)$$

Proof: Let $E_k(\lambda) = \{z \in E(\lambda) : 2^k \leq |z| < 2^{k+1}\}$, then $\forall z \in E_k(\lambda)$, $\exists r(z) > 0$, such that $\mu(B(z, r(z))) > \lambda(\frac{r(z)}{|z|})^\beta$, therefore $r(z) \leq 2^{k-1}$. Since $E_k(\lambda)$ can be covered by the union of a family of balls $\{B(z, r(z)) : z \in E_k(\lambda)\}$, by the Vitali Lemma([2]), $\exists \Lambda_k \subset E_k(\lambda)$, Λ_k is at most countable, such that $\{B(z, r(z)) : z \in \Lambda_k\}$ are disjoint and

$$E_k(\lambda) \subset \bigcup_{z \in \Lambda_k} B(z, 5r(z)),$$

so

$$E(\lambda) = \bigcup_{k=1}^{\infty} E_k(\lambda) \subset \bigcup_{k=1}^{\infty} \bigcup_{z \in \Lambda_k} B(z, 5r(z)). \quad (2.3)$$

On the other hand, note that $\bigcup_{z \in \Lambda_k} B(z, r(z)) \subset \{z : 2^{k-1} \leq |z| < 2^{k+2}\}$, so that

$$\sum_{z \in \Lambda_k} \frac{(5r(z))^\beta}{|z|^\beta} \leq 5^\beta \sum_{z \in \Lambda_k} \frac{\mu(B(z, r(z)))}{\lambda} \leq \frac{5^\beta}{\lambda} \mu\{z : 2^{k-1} \leq |z| < 2^{k+2}\}.$$

Hence we obtain

$$\sum_{k=1}^{\infty} \sum_{z \in \Lambda_k} \frac{(5r(z))^\beta}{|z|^\beta} \leq \sum_{k=1}^{\infty} \frac{5^\beta}{\lambda} \mu\{z : 2^{k-1} \leq |z| < 2^{k+2}\} \leq \frac{3\mu(\mathbf{C})5^\beta}{\lambda}. \quad (2.4)$$

Rearrange $\{z : z \in \Lambda_k, k = 1, 2, \dots\}$ and $\{5r(z) : z \in \Lambda_k, k = 1, 2, \dots\}$, we get $\{z_j\}$ and $\{\rho_j\}$ such that (2.1) and (2.2) hold.

Lemma 2 (1) $|\Im \sum_{k=0}^m \frac{z^k}{\xi^{1+k}}| \leq \sum_{k=0}^{m-1} \frac{2^k y |z|^k}{|\xi|^{2+k}};$
(2) $|\Im \sum_{k=0}^{\infty} \frac{z^{k+m+1}}{\xi^k}| \leq 2^{m+1} y |z|^m;$
(3) $|G_m(z, \zeta) - G(z, \zeta)| \leq \frac{1}{\pi} \sum_{k=1}^m \frac{ky\eta |z|^{k-1}}{|\zeta|^{1+k}};$
(4) $|G_m(z, \zeta)| \leq \frac{1}{\pi} \sum_{k=m+1}^{\infty} \frac{ky\eta |z|^{k-1}}{|\zeta|^{1+k}}.$

Now we are ready to prove Theorems.

Throughout the proof, A denote various positive constants.

Proof of Theorem 1

Define the measure $dm(\xi)$ and the kernel $K(z, \xi)$ by

$$dm(\xi) = \frac{|f(\xi)|}{1 + |\xi|^{2+m}} d\xi, \quad K(z, \xi) = P_m(z, \xi)(1 + |\xi|^{2+m}).$$

For any $\varepsilon > 0$, there exists $R_\varepsilon > 2$, such that

$$\int_{|\xi| \geq R_\varepsilon} dm(\xi) \leq \frac{\varepsilon}{5^{2-\alpha}}.$$

For every Lebesgue measurable set $E \subset \mathbf{R}$, the measure $m^{(\varepsilon)}$ defined by $m^{(\varepsilon)}(E) = m(E \cap \{x \in \mathbf{R} : |x| \geq R_\varepsilon\})$ satisfies $m^{(\varepsilon)}(\mathbf{R}) \leq \frac{\varepsilon}{5^{2-\alpha}}$, write

$$\begin{aligned} v_1(z) &= \int_{|\xi-z| \leq 3|z|} P(z, \xi)(1 + |\xi|^{2+m}) dm^{(\varepsilon)}(\xi), \\ v_2(z) &= \int_{|\xi-z| \leq 3|z|} (P_m(z, \xi) - P(z, \xi))(1 + |\xi|^{2+m}) dm^{(\varepsilon)}(\xi), \\ v_3(z) &= \int_{|\xi-z| > 3|z|} K(z, \xi) dm^{(\varepsilon)}(\xi), \\ v_4(z) &= \int_{1 < |\xi| < R_\varepsilon} K(z, \xi) dm(\xi), \\ v_5(z) &= \int_{|\xi| \leq 1} K(z, \xi) dm(\xi). \end{aligned}$$

then

$$|v(z)| \leq |v_1(z)| + |v_2(z)| + |v_3(z)| + |v_4(z)| + |v_5(z)|. \quad (2.5)$$

Let $E_1(\lambda) = \{z \in \mathbf{C} : |z| \geq 2, \exists t > 0, m^{(\varepsilon)}(B(z, t) \cap \mathbf{R}) > \lambda(\frac{t}{|z|})^{2-\alpha}\}$,
when $|z| \geq 2R_\varepsilon$, $z \notin E_1(\lambda)$, then

$$\forall t > 0, m^{(\varepsilon)}(B(z, t) \cap \mathbf{R}) \leq \lambda(\frac{t}{|z|})^{2-\alpha}.$$

So we have

$$\begin{aligned} |v_1(z)| &\leq \int_{y \leq |\xi-z| \leq 3|z|} \frac{y}{\pi|z-\xi|^2} 2|\xi|^{2+m} dm^{(\varepsilon)}(\xi) \\ &\leq \frac{2^{2m+5}}{\pi} y |z|^{2+m} \int_{y \leq |\xi-z| \leq 3|z|} \frac{1}{|z-\xi|^2} dm^{(\varepsilon)}(\xi) \\ &= \frac{2^{2m+5}}{\pi} y |z|^{m+2} \int_y^{3|z|} \frac{1}{t^2} dm_z^{(\varepsilon)}(t). \end{aligned}$$

where $m_z^{(\varepsilon)}(t) = \int_{|\xi-z| \leq t} dm^{(\varepsilon)}(\xi)$, since for $z \notin E_1(\lambda)$,

$$\begin{aligned} \int_y^{3|z|} \frac{1}{t^2} dm_z^{(\varepsilon)}(t) &\leq \frac{m_z^{(\varepsilon)}(3|z|)}{(3|z|)^2} + 2 \int_y^{3|z|} \frac{m_z^{(\varepsilon)}(t)}{t^3} dt \\ &\leq \frac{\lambda}{3^\alpha |z|^2} + 2 \int_y^{3|z|} \frac{\lambda \frac{t^{2-\alpha}}{|z|^{2-\alpha}}}{t^3} dt \\ &\leq \frac{\lambda}{|z|^2} \left[\frac{1}{3^\alpha} + \frac{2}{\alpha} \frac{|z|^\alpha}{y^\alpha} \right], \end{aligned}$$

so that

$$|v_1(z)| \leq \frac{2^{2m+5}}{\pi} \left(\frac{1}{3^\alpha} + \frac{2}{\alpha} \right) \lambda y^{1-\alpha} |z|^{m+\alpha}. \quad (2.6)$$

By (1) of Lemma 2, we obtain

$$\begin{aligned} |v_2(z)| &\leq \int_{y \leq |\xi-z| \leq 3|z|} \frac{1}{\pi} \sum_{k=0}^{m-1} \frac{2^k y |z|^k}{|\xi|^{2+k}} \cdot 2|\xi|^{2+m} dm^{(\varepsilon)}(\xi) \\ &\leq \int_{y \leq |\xi-z| \leq 3|z|} \sum_{k=0}^{m-1} \frac{2^{k+1} y |z|^k}{\pi} (4|z|)^{m-k} dm^{(\varepsilon)}(\xi) \\ &\leq \frac{2^{2m+1}}{\pi} \sum_{k=0}^{m-1} \frac{1}{2^k} \frac{1}{5^{2-\alpha}} \varepsilon y |z|^m \\ &\leq \frac{4^{m-1+\alpha}}{\pi} \varepsilon y |z|^m. \end{aligned} \quad (2.7)$$

By (2) of Lemma 2, we see that([6])

$$\begin{aligned}
|v_3(z)| &\leq \int_{|\xi-z|>3|z|} \left| \Im \sum_{k=m}^{\infty} \frac{z^{k+1}}{\pi \xi^{2+k}} \right| \cdot 2|\xi|^{2+m} dm^{(\varepsilon)}(\xi) \\
&= \int_{|\xi-z|>3|z|} \frac{2}{\pi} \left| \Im \sum_{k=0}^{\infty} \frac{z^{k+m+1}}{\xi^k} \right| dm^{(\varepsilon)}(\xi) \\
&\leq \frac{2^{m+2}}{\pi} \frac{\varepsilon}{5^{2-\alpha}} y |z|^m \\
&\leq \frac{2^{m-2+2\alpha}}{\pi} \varepsilon y |z|^m.
\end{aligned} \tag{2.8}$$

Write

$$\begin{aligned}
v_4(z) &= \int_{1<|\xi|<R_\varepsilon} [P(z, \xi) - \frac{1}{\pi} \Im \sum_{k=0}^m \frac{z^k}{\xi^{1+k}}] (1 + |\xi|^{2+m}) dm(\xi) \\
&= v_{41}(z) - v_{42}(z),
\end{aligned}$$

then

$$\begin{aligned}
|v_{41}(z)| &\leq \int_{1<|\xi|<R_\varepsilon} \frac{y}{\pi |z - \xi|^2} 2|\xi|^{2+m} dm(\xi) \\
&\leq \frac{2R_\varepsilon^{2+m}y}{\pi} \int_{1<|\xi|<R_\varepsilon} \frac{1}{(\frac{|z|}{2})^2} dm(\xi) \\
&\leq \frac{2^3 R_\varepsilon^{2+m} m(\mathbf{R})}{\pi} \frac{y}{|z|^2}.
\end{aligned} \tag{2.9}$$

by (1) of Lemma 2, we obtain

$$\begin{aligned}
|v_{42}(z)| &\leq \int_{1<|\xi|<R_\varepsilon} \frac{1}{\pi} \sum_{k=0}^{m-1} \frac{2^k y |z|^k}{|\xi|^{2+k}} \cdot 2|\xi|^{2+m} dm(\xi) \\
&\leq \sum_{k=0}^{m-1} \frac{2^{k+1}}{\pi} y |z|^k R_\varepsilon^{m-k} m(\mathbf{R}) \\
&\leq \frac{2^{m+1} R_\varepsilon^m m(\mathbf{R})}{\pi} y |z|^{m-1}.
\end{aligned} \tag{2.10}$$

In case $|\xi| \leq 1$, note that

$$K(z, \xi) = P_m(z, \xi)(1 + |\xi|^{2+m}) \leq \frac{2y}{\pi |z - \xi|^2},$$

so that

$$|v_5(z)| \leq \int_{|\xi| \leq 1} \frac{2y}{\pi (\frac{|z|}{2})^2} dm(\xi) \leq \frac{2^3 m(\mathbf{R})}{\pi} \frac{y}{|z|^2}. \tag{2.11}$$

Thus, by collecting (2.5), (2.6), (2.7), (2.8), (2.9), (2.10) and (2.11), there exists a positive constant A independent of ε , such that if $|z| \geq 2R_\varepsilon$ and $z \notin E_1(\varepsilon)$, we have

$$|v(z)| \leq A\varepsilon y^{1-\alpha} |z|^{m+\alpha}.$$

Let μ_ε be a measure in \mathbf{C} defined by $\mu_\varepsilon(E) = m^{(\varepsilon)}(E \cap \mathbf{R})$ for every measurable set E in \mathbf{C} . Take $\varepsilon = \varepsilon_p = \frac{1}{2^{p+2}}$, $p = 1, 2, 3, \dots$, then there exists a sequence $\{R_p\}$: $1 = R_0 < R_1 < R_2 < \dots$ such that

$$\mu_{\varepsilon_p}(\mathbf{C}) = \int_{|\xi| \geq R_p} dm(\xi) < \frac{\varepsilon_p}{5^{2-\alpha}}.$$

Take $\lambda = 3 \cdot 5^{2-\alpha} \cdot 2^p \mu_{\varepsilon_p}(\mathbf{C})$ in Lemma 1, then $\exists z_{j,p}$ and $\rho_{j,p}$, where $R_{p-1} \leq |z_{j,p}| < R_p$ such that

$$\sum_{j=1}^{\infty} \left(\frac{\rho_{j,p}}{|z_{j,p}|} \right)^{2-\alpha} \leq \frac{1}{2^p}.$$

So if $R_{p-1} \leq |z| < R_p$, $z \notin G_p = \cup_{j=1}^{\infty} B(z_{j,p}, \rho_{j,p})$, we have

$$|v(z)| \leq A\varepsilon_p y^{1-\alpha} |z|^{m+\alpha},$$

thereby

$$\sum_{p=1}^{\infty} \sum_{j=1}^{\infty} \left(\frac{\rho_{j,p}}{|z_{j,p}|} \right)^{2-\alpha} \leq \sum_{p=1}^{\infty} \frac{1}{2^p} = 1 < \infty.$$

Set $G = \cup_{p=1}^{\infty} G_p$, then Theorem 1 holds.

Proof of Theorem 2

Define the measure $dn(\zeta)$ and the kernel $L(z, \zeta)$ by

$$dn(\zeta) = \frac{\eta d\mu(\zeta)}{1 + |\zeta|^{2+m}}, \quad L(z, \zeta) = G_m(z, \zeta) \frac{1 + |\zeta|^{2+m}}{\eta}.$$

then the function $h(z)$ can be written as

$$h(z) = \int_{\mathbf{C}_+} L(z, \zeta) dn(\zeta).$$

For any $\varepsilon > 0$, there exists $R_\varepsilon > 2$, such that

$$\int_{|\zeta| \geq R_\varepsilon} dn(\zeta) < \frac{\varepsilon}{5^{2-\alpha}}.$$

For every Lebesgue measurable set $E \subset \mathbf{C}$, the measure $n^{(\varepsilon)}$ defined by $n^{(\varepsilon)}(E) = n(E \cap \{\zeta \in \mathbf{C}_+ : |\zeta| \geq R_\varepsilon\})$ satisfies $n^{(\varepsilon)}(\mathbf{C}_+) \leq \frac{\varepsilon}{5^{2-\alpha}}$,

write

$$\begin{aligned}
h_1(z) &= \int_{|\zeta-z| \leq \frac{y}{2}} G(z, \zeta) \frac{1 + |\zeta|^{2+m}}{\eta} dn^{(\varepsilon)}(\zeta), \\
h_2(z) &= \int_{\frac{y}{2} < |\zeta-z| \leq 3|z|} G(z, \zeta) \frac{1 + |\zeta|^{2+m}}{\eta} dn^{(\varepsilon)}(\zeta), \\
h_3(z) &= \int_{|\zeta-z| \leq 3|z|} (G_m(z, \zeta) - G(z, \zeta)) \frac{1 + |\zeta|^{2+m}}{\eta} dn^{(\varepsilon)}(\zeta), \\
h_4(z) &= \int_{|\zeta-z| > 3|z|} L(z, \zeta) dn^{(\varepsilon)}(\zeta), \\
h_5(z) &= \int_{1 < |\zeta| < R_\varepsilon} L(z, \zeta) dn(\zeta), \\
h_6(z) &= \int_{|\zeta| \leq 1} L(z, \zeta) dn(\zeta).
\end{aligned}$$

then

$$h(z) = h_1(z) + h_2(z) + h_3(z) + h_4(z) + h_5(z) + h_6(z). \quad (2.12)$$

Let $E_2(\lambda) = \{z \in \mathbf{C} : |z| \geq 2, \exists t > 0, n^{(\varepsilon)}(B(z, t) \cap \mathbf{C}_+) > \lambda(\frac{t}{|z|})^{2-\alpha}\}$,
when $|z| \geq 2R_\varepsilon$, $z \notin E_2(\lambda)$, then

$$\forall t > 0, n^{(\varepsilon)}(B(z, t) \cap \mathbf{C}_+) \leq \lambda(\frac{t}{|z|})^{2-\alpha}.$$

So we have

$$\begin{aligned}
|h_1(z)| &\leq \int_{|\zeta-z| \leq \frac{y}{2}} \frac{1}{2\pi} \log \left| \frac{\zeta - \bar{z}}{\zeta - z} \right| \frac{1 + |\zeta|^{2+m}}{\eta} dn^{(\varepsilon)}(\zeta) \\
&\leq \int_{|\zeta-z| \leq \frac{y}{2}} \frac{1}{2\pi} \log \frac{3y}{|\zeta - z|} \frac{2|\zeta|^{2+m}}{\frac{y}{2}} dn^{(\varepsilon)}(\zeta) \\
&\leq \frac{2 \times (3/2)^{2+m}}{\pi} \frac{|z|^{2+m}}{y} \int_{|\zeta-z| \leq \frac{y}{2}} \log \frac{3y}{|\zeta - z|} dn^{(\varepsilon)}(\zeta) \\
&= \frac{2 \times (3/2)^{2+m}}{\pi} \frac{|z|^{2+m}}{y} \int_0^{\frac{y}{2}} \log \frac{3y}{t} dn_z^{(\varepsilon)}(t) \\
&\leq \frac{2 \times (3/2)^{2+m}}{\pi} \left[\frac{\log 6}{2^{2-\alpha}} + \frac{1}{(2-\alpha)2^{2-\alpha}} \right] \lambda y^{1-\alpha} |z|^{m+\alpha}. \quad (2.13)
\end{aligned}$$

where $n_z^{(\varepsilon)}(t) = \int_{|\zeta-z| \leq t} dn^{(\varepsilon)}(\zeta)$.

Note that

$$|G(z, \zeta)| = |E(z - \zeta) - E(z - \bar{\zeta})| \leq \frac{y\eta}{\pi|z - \zeta|^2} \quad (2.14)$$

then by (2.14), we have

$$\begin{aligned}
|h_2(z)| &\leq \int_{\frac{y}{2} < |\zeta - z| \leq 3|z|} \frac{y\eta}{\pi|z - \zeta|^2} \frac{2|\zeta|^{2+m}}{\eta} dn^{(\varepsilon)}(\zeta) \\
&\leq \frac{2^{2m+5}}{\pi} y|z|^{2+m} \int_{\frac{y}{2} < |\zeta - z| \leq 3|z|} \frac{1}{|z - \zeta|^2} dn^{(\varepsilon)}(\zeta) \\
&= \frac{2^{2m+5}}{\pi} y|z|^{2+m} \int_{\frac{y}{2}}^{3|z|} \frac{1}{t^2} dn_z^{(\varepsilon)}(t) \\
&\leq \frac{2^{2m+5}}{\pi} y|z|^{2+m} \frac{\lambda}{|z|^2} \left(\frac{1}{3^\alpha} + \frac{2^{\alpha+1}}{\alpha} \frac{|z|^\alpha}{y^\alpha} \right) \\
&\leq \frac{2^{2m+5}}{\pi} \left(\frac{1}{3^\alpha} + \frac{2^{\alpha+1}}{\alpha} \right) \lambda y^{1-\alpha} |z|^{m+\alpha}.
\end{aligned} \tag{2.15}$$

By (3) of Lemma 2, we obtain

$$\begin{aligned}
|h_3(z)| &\leq \int_{|\zeta - z| \leq 3|z|} \frac{1}{\pi} \sum_{k=1}^m \frac{ky\eta|z|^{k-1}}{|\zeta|^{1+k}} \frac{2|\zeta|^{2+m}}{\eta} dn^{(\varepsilon)}(\zeta) \\
&\leq \int_{|\zeta - z| \leq 3|z|} \frac{2}{\pi} \sum_{k=1}^m ky|z|^{k-1} (4|z|)^{m-k+1} dn^{(\varepsilon)}(\zeta) \\
&\leq \frac{2^{2m+1}}{\pi} \sum_{k=1}^m \frac{k}{4^{k-1}} \frac{1}{5^{2-\alpha}} \varepsilon y |z|^m \\
&\leq \frac{2^{2m+2\alpha+1}}{9\pi} \varepsilon y |z|^m.
\end{aligned} \tag{2.16}$$

By (4) of Lemma 2, we see that

$$\begin{aligned}
|h_4(z)| &\leq \int_{|\zeta - z| > 3|z|} \frac{1}{\pi} \sum_{k=m+1}^{\infty} \frac{ky\eta|z|^{k-1}}{|\zeta|^{1+k}} \frac{2|\zeta|^{2+m}}{\eta} dn^{(\varepsilon)}(\zeta) \\
&\leq \int_{|\zeta - z| > 3|z|} \frac{2}{\pi} \sum_{k=m+1}^{\infty} ky \frac{|z|^{k-1}}{(2|z|)^{k-m-1}} dn^{(\varepsilon)}(\zeta) \\
&\leq \frac{2^{m+2}}{\pi} \sum_{k=m+1}^{\infty} \frac{k}{2^k} \frac{1}{5^{2-\alpha}} \varepsilon y |z|^m \\
&\leq \frac{4^{\alpha-1}(m+2)}{\pi} \varepsilon y |z|^m.
\end{aligned} \tag{2.17}$$

Write

$$\begin{aligned}
h_5(z) &= \int_{1 < |\zeta| < R_\varepsilon} [G(z, \zeta) + (G_m(z, \zeta) - G(z, \zeta))] \frac{1 + |\zeta|^{2+m}}{\eta} dn(\zeta) \\
&= h_{51}(z) + h_{52}(z),
\end{aligned}$$

then we obtain by (2.14)

$$\begin{aligned}
|h_{51}(z)| &\leq \int_{1 < |\zeta| < R_\varepsilon} \frac{y\eta}{\pi|z - \zeta|^2} \frac{2|\zeta|^{2+m}}{\eta} dn(\zeta) \\
&\leq \frac{2R_\varepsilon^{2+m}}{\pi} y \int_{1 < |\zeta| < R_\varepsilon} \frac{1}{\left(\frac{|z|}{2}\right)^2} dn(\zeta) \\
&\leq \frac{2^3 R_\varepsilon^{2+m} n(\mathbf{C}_+)}{\pi} \frac{y}{|z|^2}.
\end{aligned} \tag{2.18}$$

by (3) of Lemma 2, we obtain

$$\begin{aligned}
|h_{52}(z)| &\leq \int_{1 < |\zeta| < R_\varepsilon} \frac{1}{\pi} \sum_{k=1}^m \frac{ky\eta|z|^{k-1}}{|\zeta|^{1+k}} \frac{2|\zeta|^{2+m}}{\eta} dn(\zeta) \\
&\leq \frac{2}{\pi} \sum_{k=1}^m ky|z|^{k-1} R_\varepsilon^{m-k+1} n(\mathbf{C}_+) \\
&\leq \frac{m(m+1)R_\varepsilon^m n(\mathbf{C}_+)}{\pi} y|z|^{m-1}.
\end{aligned} \tag{2.19}$$

In case $|\zeta| \leq 1$, by (2.14), we have

$$|L(z, \zeta)| \leq \frac{y\eta}{\pi|z - \zeta|^2} \frac{2}{\eta} = \frac{2y}{\pi|z - \zeta|^2},$$

so that

$$|h_6(z)| \leq \int_{|\zeta| \leq 1} \frac{2y}{\pi\left(\frac{|z|}{2}\right)^2} dn(\zeta) \leq \frac{2^3 n(\mathbf{C}_+)}{\pi} \frac{y}{|z|^2}. \tag{2.20}$$

Thus, by collecting (2.12), (2.13), (2.15), (2.16), (2.17), (2.18), (2.19) and (2.20), there exists a positive constant A independent of ε , such that if $|z| \geq 2R_\varepsilon$ and $z \notin E_2(\varepsilon)$, we have

$$|h(z)| \leq A\varepsilon y^{1-\alpha} |z|^{m+\alpha}.$$

Similarly, if $z \notin G$, we have

$$h(z) = o(y^{1-\alpha} |z|^{m+\alpha}) \quad \text{as } |z| \rightarrow \infty. \tag{2.21}$$

by (1.9) and (2.21), we obtain

$$u(z) = v(z) + h(z) = o(y^{1-\alpha} |z|^{m+\alpha}) \quad \text{as } |z| \rightarrow \infty$$

hold in $\mathbf{C}_+ - G$.

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¹ SCH. MATH. SCI. & LAB. MATH. COM. SYS., BEIJING NORMAL UNIVERSITY, 100875 BEIJING, THE PEOPLE'S REPUBLIC OF CHINA

² DEPARTMENT OF PUBLIC BASIC COURSES, BEIJING INSTITUTE OF FASHION AND TECHNOLOGY, 100029 BEIJING, THE PEOPLE'S REPUBLIC OF CHINA
E-mail address: denggt@bnu.edu.cn